RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS

4727

Further Pure Mathematics 3

Candidates answer on the answer booklet.

OCR supplied materials:
- 8 page answer booklet (sent with general stationery)
- List of Formulae (MF1)

Other materials required:
- Scientific or graphical calculator

Monday 13 June 2011
Morning

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet. Please write clearly and in capital letters.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a scientific or graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

1 A line l has equation $\frac{x-1}{5} = \frac{y-6}{6} = \frac{z+3}{-7}$ and a plane p has equation $x + 2y - z = 40$.

(i) Find the acute angle between l and p. **[4]**

(ii) Find the perpendicular distance from the point $(1, 6, -3)$ to p. **[2]**

2 It is given that $z = e^{i\theta}$, where $0 < \theta < 2\pi$, and $w = \frac{1+z}{1-z}$.

(i) Prove that $w = i\cot\frac{1}{2}\theta$. **[3]**

(ii) Sketch separate Argand diagrams to show the locus of z and the locus of w. You should show the direction in which each locus is described when θ increases in the interval $0 < \theta < 2\pi$. **[3]**

3 The variables x and y satisfy the differential equation

$$\frac{dy}{dx} + 4y = 5\cos 3x.$$

(i) Find the complementary function. **[2]**

(ii) Hence, or otherwise, find the general solution. **[7]**

(iii) Find the approximate range of values of y when x is large and positive. **[2]**

4 A group G, of order 8, is generated by the elements a, b, c. G has the properties

$$a^2 = b^2 = c^2 = e, \quad ab = ba, \quad bc = cb, \quad ca = ac,$$

where e is the identity.

(i) Using these properties and basic group properties as necessary, prove that $abc = cba$. **[2]**

The operation table for G is shown below.

	e	a	b	c	bc	ca	ab	abc
e	e	a	b	c	bc	ca	ab	abc
a	a	e	ab	ca	abc	c	b	bc
b	b	ab	e	bc	c	abc	a	ca
c	c	ca	bc	e	b	a	abc	ab
bc	bc	abc	c	b	e	ab	ca	a
ca	ca	c	abc	a	ab	e	bc	b
ab	ab	b	a	abc	ca	bc	e	c
abc	abc	bc	ca	ab	a	b	c	e

(ii) List all the subgroups of order 2. **[2]**

(iii) List five subgroups of order 4. **[3]**

(iv) Determine whether all the subgroups of G which are of order 4 are isomorphic. **[2]**

5 The substitution $y = u^k$, where k is an integer, is to be used to solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = x^2y^2 \qquad \text{(A)}$$

by changing it into an equation (B) in the variables u and x.

(i) Show that equation (B) may be written in the form

$$\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{3}{kx}u = \frac{1}{k}xu^{k+1}.$$ **[4]**

(ii) Write down the value of k for which the integrating factor method may be used to solve equation (B). **[1]**

(iii) Using this value of k, solve equation (B) and hence find the general solution of equation (A), giving your answer in the form $y = \mathrm{f}(x)$. **[4]**

6 **(a)** The set of polynomials $\{ax + b\}$, where $a, b \in \mathbb{R}$, is denoted by P. Assuming that the associativity property holds, prove that P, under addition, is a group. **[4]**

(b) The set of polynomials $\{ax + b\}$, where $a, b \in \{0, 1, 2\}$, is denoted by Q. It is given that Q, under addition modulo 3, is a group, denoted by $\big(Q, +(\text{mod}3)\big)$.

(i) State the order of the group. **[1]**

(ii) Write down the inverse of the element $2x + 1$. **[1]**

(iii) $\mathrm{q}(x) = ax + b$ is any element of Q other than the identity. Find the order of $\mathrm{q}(x)$ and hence determine whether $\big(Q, +(\text{mod}3)\big)$ is a cyclic group. **[4]**

7 (In this question, the notation ΔABC denotes the area of the triangle ABC.)

The points P, Q and R have position vectors $p\mathbf{i}$, $q\mathbf{j}$ and $r\mathbf{k}$ respectively, relative to the origin O, where p, q and r are positive. The points O, P, Q and R are joined to form a tetrahedron.

(i) Draw a sketch of the tetrahedron and write down the values of ΔOPQ, ΔOQR and ΔORP. **[3]**

(ii) Use the definition of the vector product to show that $\frac{1}{2}\left|\overrightarrow{RP} \times \overrightarrow{RQ}\right| = \Delta PQR$. **[1]**

(iii) Show that $(\Delta OPQ)^2 + (\Delta OQR)^2 + (\Delta ORP)^2 = (\Delta PQR)^2$. **[6]**

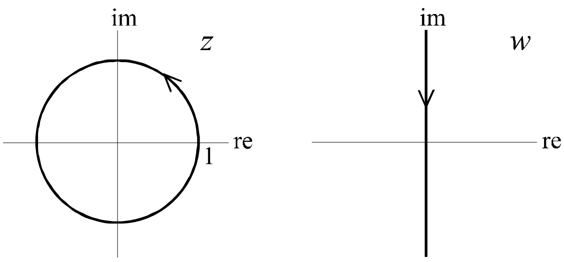
8 **(i)** Use de Moivre's theorem to express $\cos 4\theta$ as a polynomial in $\cos\theta$. **[4]**

(ii) Hence prove that $\cos 4\theta \cos 2\theta \equiv 16\cos^6\theta - 24\cos^4\theta + 10\cos^2\theta - 1$. **[1]**

(iii) Use part **(ii)** to show that the only roots of the equation $\cos 4\theta \cos 2\theta = 1$ are $\theta = n\pi$, where n is an integer. **[3]**

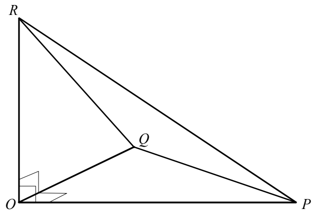
(iv) Show that $\cos 4\theta \cos 2\theta = -1$ only when $\cos\theta = 0$. **[3]**

1 (i)	$\theta = \sin^{-1} \dfrac{\lvert[5, 6, -7] . [1, 2, -1]\rvert}{\sqrt{5^2+6^2+(-7)^2}\sqrt{1^2+2^2+(-1)^2}}$	M1*	For using scalar product of line and plane vectors
		M1 (*dep)	For both moduli seen
	$\theta = \sin^{-1} \dfrac{24}{\sqrt{110}\sqrt{6}} = 69.1^\circ$ $(69.099...^\circ, 1.206)$	A1	For correct scalar product
		A1 **4**	For correct angle
	$\phi = \sin^{-1} \dfrac{\lvert[5, 6, -7] \times [1, 2, -1]\rvert}{\sqrt{5^2+6^2+(-7)^2}\sqrt{1^2+2^2+(-1)^2}}$	**SR** M1*	For vector product of line and plane vectors AND finding modulus of result
		M1 (*dep)	For moduli of line and plane vectors seen
	$\phi = \sin^{-1} \dfrac{\sqrt{84}}{\sqrt{110}\sqrt{6}} = 20.9^\circ \Rightarrow \theta = 69.1^\circ$	A1	For correct modulus $\sqrt{84}$
		A1	For correct angle
(ii)	METHOD 1		
	$d = \dfrac{\lvert 1+12+3-40\rvert}{\sqrt{1^2+2^2+(-1)^2}} = \dfrac{24}{\sqrt{6}} = 4\sqrt{6} \approx 9.80$	M1	For use of correct formula
		A1 **2**	For correct distance
	METHOD 2		
	$(1+\lambda)+2(6+2\lambda)-(-3-\lambda)=40$	M1	For substituting parametric form into plane
	$\Rightarrow \lambda = 4 \Rightarrow d = 4\sqrt{6}$	A1	For correct distance
	OR distance from $(1, 6, -3)$ to $(5, 14, -7)$ $= \sqrt{4^2+8^2+(-4)^2} = \sqrt{96}$		
	METHOD 3		
	Plane through $(1, 6, -3)$ parallel to p is	M1	For finding parallel plane through $(1, 6, -3)$
	$x+2y-z=16 \Rightarrow d = \dfrac{40-16}{\sqrt{6}} = \dfrac{24}{\sqrt{6}}$	A1	For correct distance
	METHOD 4		
	e.g. $(0, 0, -40)$ on p $\Rightarrow$ vector to $(1, 6, -3)$ $= \pm(1, 6, 37)$	M1	For using any point on p to find vector and scalar product seen e.g. $[1, 6, 37] . [1, 2, -1]$
	$d = \dfrac{\lvert[1, 6, 37] . [1, 2, -1]\rvert}{\sqrt{6}} = \dfrac{24}{\sqrt{6}}$	A1	For correct distance
	METHOD 5		
	l meets p where $(1+5t)+2(6+6t)-(-3-7t)=40$ $\Rightarrow t=1 \Rightarrow d = \lvert[5, 6, -7]\rvert \sin\theta$	M1	For finding t where l meets p and linking d with triangle
	$\Rightarrow d = \sqrt{110}\dfrac{24}{\sqrt{110}\sqrt{6}} = \dfrac{24}{\sqrt{6}}$	A1	For correct distance
		6	
2 (i)	METHOD 1		
	EITHER $\dfrac{1+e^{i\theta}}{1-e^{i\theta}} = \dfrac{e^{-\frac{1}{2}i\theta}+e^{\frac{1}{2}i\theta}}{e^{-\frac{1}{2}i\theta}-e^{\frac{1}{2}i\theta}}$	M1	*EITHER* For changing LHS terms to $e^{\pm\frac{1}{2}i\theta}$ *OR in reverse* For using $\cot\frac{1}{2}\theta = \dfrac{\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}$
	$= \dfrac{2\cos\frac{1}{2}\theta}{-2i\sin\frac{1}{2}\theta} = i\cot\frac{1}{2}\theta$	M1	For either of $\dfrac{\cos}{\sin}\frac{1}{2}\theta = \dfrac{e^{\frac{1}{2}i\theta} \pm e^{-\frac{1}{2}i\theta}}{(2)(i)}$ soi
	OR in reverse with similar working	A1 **3**	For fully correct proof to **AG** **SR** If factors of 2 or i are not clearly seen, award M1 M1 A0

Question	Answer	Marks		Guidance
2 (i)	METHOD 2			
	EITHER $\frac{1+e^{i\theta}}{1-e^{i\theta}} \times \frac{1-e^{-i\theta}}{1-e^{-i\theta}} = \frac{e^{i\theta}-e^{-i\theta}}{2-\left(e^{i\theta}+e^{-i\theta}\right)}$	M1		For multiplying top and bottom by complex conjugate in exp or trig form
	OR $\frac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta} \times \frac{1-\cos\theta+i\sin\theta}{1-\cos\theta+i\sin\theta}$			
	$= \frac{2i\sin\theta}{2-2\cos\theta} = \frac{2i\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta} = i\cot\frac{1}{2}\theta$	M1 A1		For using both double angle formulae correctly For fully correct proof to **AG**
	METHOD 3			
	$\frac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta} = \frac{2\cos^2\frac{1}{2}\theta+2i\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta-2i\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}$	M1		For using both double angle formulae correctly
	$= \frac{2\cos\frac{1}{2}\theta\left(\cos\frac{1}{2}\theta+i\sin\frac{1}{2}\theta\right)}{2\sin\frac{1}{2}\theta\left(\sin\frac{1}{2}\theta-i\cos\frac{1}{2}\theta\right)}$	M1		For appropriate factorisation
	$= i\cot\frac{1}{2}\theta\frac{\left(\sin\frac{1}{2}\theta-i\cos\frac{1}{2}\theta\right)}{\left(\sin\frac{1}{2}\theta-i\cos\frac{1}{2}\theta\right)} = i\cot\frac{1}{2}\theta$	A1		For fully correct proof to **AG**
	METHOD 4			
	$\frac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta} = \frac{1+\frac{1-t^2}{1+t^2}+i\frac{2t}{1+t^2}}{1-\frac{1-t^2}{1+t^2}-i\frac{2t}{1+t^2}}$	M1		For substituting both *t* formulae correctly
	$= \frac{2+2it}{2t^2-2it} = \frac{1}{t}\frac{1+it}{t-i} = \frac{i}{t}\frac{t-i}{t-i} = i\cot\frac{1}{2}\theta$	M1 A1		For appropriate factorisation For fully correct proof to **AG**
	METHOD 5			
	$\frac{1+e^{i\theta}}{1-e^{i\theta}} \times \frac{1+e^{i\theta}}{1+e^{i\theta}} = \frac{1+2e^{i\theta}+e^{2i\theta}}{1-e^{2i\theta}}$ $= \frac{2+e^{i\theta}+e^{-i\theta}}{e^{-i\theta}-e^{i\theta}}$	M1		For multiplying top and bottom by $1+e^{i\theta}$ and attempting to divide by $e^{i\theta}$ *OR* multiplying top and bottom by $1+e^{-i\theta}$
	$= \frac{2(1+\cos\theta)}{-2i\sin\theta} = \frac{2\cos^2\frac{1}{2}\theta}{-2i\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} = \frac{\cos\frac{1}{2}\theta}{-i\sin\frac{1}{2}\theta}$	M1		For using both double angle formulae correctly
	$= i\cot\frac{1}{2}\theta$	A1	**3**	For fully correct proof to **AG**
(ii)		M1 A1 B1	 **3**	For a circle centre *O* For indication of radius = 1 and anticlockwise arrow shown For locus of *w* shown as imaginary axis described downwards
			6	

3	(i)	METHOD 1 $m+4\ (=0)\ \Rightarrow$ CF $(y=)Ae^{-4x}$	M1 A1	 2	For correct auxiliary equation (soi) For correct CF
		METHOD 2 Separating variables on $\frac{dy}{dx}+4y=0$ $\Rightarrow \ln y=-4x$ $\Rightarrow$ CF $(y=)Ae^{-4x}$	 M1 A1		 For integration to this stage For correct CF
	(ii)	PI $(y=)\ p\cos 3x+q\sin 3x$ $y'=-3p\sin 3x+3q\cos 3x$ $\Rightarrow (-3p+4q)\sin 3x+(4p+3q)\cos 3x=5\cos 3x$ $\Rightarrow \left.\begin{array}{r}-3p+4q=0\\ 4p+3q=5\end{array}\right\} \Rightarrow p=\frac{4}{5},\ q=\frac{3}{5}$ GS $(y=)\ Ae^{-4x}+\frac{4}{5}\cos 3x+\frac{3}{5}\sin 3x$	B1 M1 A1 M1 A1 A1 B1√	 7	For stating PI of correct form For substituting y and y' into DE For correct equation For equating coeffs and solving For correct value of p, and of q For GS f.t. from their CF+PI with 1 arbitrary constant in CF and none in PI
		SR Integrating factor method may be used, followed by 2-stage integration by parts or C+iS method Marks for **(i)** are awarded only if CF is clearly identified			
	(iii)	$e^{-4x}\to 0$, $\frac{4}{5}\cos 3x+\frac{3}{5}\sin 3x=\frac{\sin}{\cos}(3x+\alpha)$ $\Rightarrow\ -1\leqslant y\leqslant 1\quad OR\quad -1\lessapprox y\lessapprox 1$	M1 A1√	 2	For considering either term For correct range (allow <) **CWO** f.t. as $-\sqrt{p^2+q^2}\leqslant y\leqslant\sqrt{p^2+q^2}$ from **(ii)**
				11	
4	(i)	$abc=(ab)c=(ba)c=b(ac)=$ $b(ca)=(bc)a=(cb)a=cba$ Minimum working: $abc=bac=bca=cba$ $OR\ abc=acb=cab=cba$ $OR\ abc=bac=bca=cba$	M1 A1	 2	For using commutativity correctly For correct proof (use of associativity may be implied)
	(ii)	$\{e,a\},\{e,b\},\{e,c\},\{e,bc\},\{e,ca\},\{e,ab\},\{e,abc\}$	B1 B1	 2	For any 5 subgroups For the other 2 subgroups and none incorrect
	(iii)	$\{e,a,b,ab\}$, $\{e,a,c,ca\}$, $\{e,b,c,bc\}$ $\{e,a,bc,abc\}$, $\{e,b,ca,abc\}$, $\{e,c,ab,abc\}$ $\{e,bc,ca,ab\}$	B1 B1 B1	 3	For any 3 subgroups For 1 more subgroup For 1 more subgroup (5 in total) and none incorrect
	(iv)	All elements $(\neq e)$ have order 2 *OR* all are self-inverse *OR* no element of G has order 4 *OR* no order 4 subgroup has a generator *or* is cyclic *OR* subgroups are of the form $\{e,a,b,ab\}$ (the Klein group) $\Rightarrow$ all order 4 subgroups are isomorphic	B1* B1 (*dep)	 2	For appropriate reference to order of elements in G For correct conclusion
				9	

5	(i)	$\frac{dy}{dx} = k u^{k-1} \frac{du}{dx}$	M1 A1		For using chain rule For correct $\frac{dy}{dx}$
		$\Rightarrow x k u^{k-1} \frac{du}{dx} + 3u^k = x^2 u^{2k}$	M1		For substituting for y and $\frac{dy}{dx}$
		$\Rightarrow \frac{du}{dx} + \frac{3}{kx} u = \frac{1}{k} x u^{k+1}$	A1	4	For correct equation **AG**
	(ii)	$k = -1$	B1	1	For correct k
	(iii)	$\frac{du}{dx} - \frac{3}{x} u = -x \Rightarrow$ IF $e^{-\int \frac{3}{x} dx} = e^{-3\ln x} = \frac{1}{x^3}$	B1√		For correct IF f.t. for IF = $x^{\frac{3}{k}}$ using k or their numerical value for k
		$\Rightarrow \frac{d}{dx}\left(u.\frac{1}{x^3}\right) = -\frac{1}{x^2}$	M1		For $\frac{d}{dx}(u.\text{their IF}) = -x.\text{their IF}$
		$\Rightarrow u.\frac{1}{x^3} = \frac{1}{x} (+c) \Rightarrow y = \frac{1}{cx^3 + x^2}$	A1 A1	 4	For correct integration both sides For correct solution for y
				9	
6	(a)	Closure $(ax+b)+(cx+d) = (a+c)x+(b+d)$	B1		For obtaining correct sum from 2 distinct elements
		$\in P$	B1		For stating result is in P *OR* is of the correct form **SR** award this mark if any of the closure result, the identity or the inverse element is stated to be in P *OR* of the correct form
		Identity $0x+0$	B1		For stating identity (allow 0)
		Inverse $-ax-b$	B1	4	For stating inverse
	(b) (i)	Order 9	B1*	1	For correct order
	(ii)	$x+2$	B1	1	For correct inverse element
	(iii)	$(ax+b)+(ax+b)+(ax+b) = 3ax+3b$	M1		For considering sums of $ax+b$ and obtaining $3ax+3b$
		$= 0x+0$ $\Rightarrow ax+b$ has order 3 $\forall\ a, b$ (except $a=b=0$)	A1		For equating to $0x+0$ *OR* 0 and obtaining order 3 **SR** For order 3 stated only *OR* found from incomplete consideration of numerical cases award B1
		Cyclic group of order 9 has element(s) of order 9	M1 (*dep)		For reference to element(s) of order 9
		$\Rightarrow (Q, +(\text{mod}\,3))$ is not cyclic	A1	4	For correct conclusion
				10	

Question	Answer	Marks		Guidance
7 (i)		B1		For sketch of tetrahedron labelled in some way At least one right angle at O must be indicated or clearly implied
		M1		For using $\Delta = \frac{1}{2}$ base × height
	$\Delta\, OPQ = \frac{1}{2}pq$, $\Delta\, OQR = \frac{1}{2}qr$, $\Delta\, ORP = \frac{1}{2}rp$	A1	3	For all areas correct **CAO**
(ii)	$\frac{1}{2}\left\lvert \overrightarrow{RP} \times \overrightarrow{RQ} \right\rvert = \frac{1}{2} \lvert \overrightarrow{RP} \rvert \lvert \overrightarrow{RQ} \rvert \sin R = \Delta\, PQR$	B1	1	For correct justification
(iii)	LHS = $\left(\frac{1}{2}pq\right)^2 + \left(\frac{1}{2}qr\right)^2 + \left(\frac{1}{2}rp\right)^2$	B1		For correct expression
	$\Delta\, PQR = \frac{1}{2}\lvert (p\mathbf{i} - q\mathbf{j}) \times (p\mathbf{i} - r\mathbf{k}) \rvert$ *OR* $\frac{1}{2}\lvert (p\mathbf{i} - r\mathbf{k}) \times (q\mathbf{j} - r\mathbf{k}) \rvert$ *OR* $\frac{1}{2}\lvert (p\mathbf{i} - q\mathbf{j}) \times (q\mathbf{j} - r\mathbf{k}) \rvert$	B1		For ΔPQR in vector form
	$\Delta\, PQR = \frac{1}{2}\lvert qr\mathbf{i} + pr\mathbf{j} + pq\mathbf{k} \rvert$	M1		For finding vector product of their attempt at ΔPQR
		A1		For correct expression
	RHS $= \frac{1}{4}\left((pq)^2 + (qr)^2 + (rp)^2\right)$	M1		For using $\lvert a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \rvert = \sqrt{a^2 + b^2 + c^2}$
		A1	6	For completing proof of **AG WWW**
			10	

8 (i)	$\text{Re}(c+\mathrm{i}s)^4 = \cos 4\theta = c^4 - 6c^2s^2 + s^4$	M1*		For expanding $(c+\mathrm{i}s)^4$: at least 2 terms and 1 binomial coefficient needed
		A1		For 3 correct terms
	$\cos 4\theta = c^4 - 6c^2(1-c^2) + (1-c^2)^2$	M1 (*dep)		For using $s^2 = 1-c^2$
	$\Rightarrow \cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$	A1	**4**	For correct expression for $\cos 4\theta$ **CAO**
(ii)	$\cos 4\theta \cos 2\theta = \left(8c^4 - 8c^2 + 1\right)\left(2c^2 - 1\right)$			For multiplying by $\left(2c^2-1\right)$
	$= 16\cos^6\theta - 24\cos^4\theta + 10\cos^2\theta - 1$	B1	**1**	to obtain **AG WWW**
(iii)	$16c^6 - 24c^4 + 10c^2 - 2 = 0$ $\Rightarrow \left(c^2-1\right)\left(8c^4 - 4c^2 + 1\right) = 0$	M1		For factorising sextic with $(c-1), (c+1)$ *or* (c^2-1)
	For quartic, $b^2 - 4ac = 16 - 32 < 0$	A1		For justifying no other roots **CWO**
	$\Rightarrow c = \pm 1$ only $\Rightarrow \theta = n\pi$	A1	**3**	For obtaining $\theta = n\pi$ **AG** Note that M1 A0 A1 is possible
			SR	For verifying $\theta = n\pi$ by substituting $c = \pm 1$ into $16c^6 - 24c^4 + 10c^2 - 2 = 0$ B1
(iv)	$16c^6 - 24c^4 + 10c^2 = 0$ $\Rightarrow c^2\left(8c^4 - 12c^2 + 5\right) = 0$	M1		For factorising sextic with c^2
	For quartic, $b^2 - 4ac = 144 - 160 < 0$	A1		For justifying no other roots **CWO**
	$\Rightarrow \cos\theta = 0$ only	A1	**3**	For correct condition obtained **AG** Note that M1 A0 A1 is possible
			SR	For verifying $\cos\theta = 0$ by substituting $c = 0$ into $16c^6 - 24c^4 + 10c^2 = 0$ B1
			SR	For verifying $\theta = \frac{1}{2}\pi$ and $\theta = -\frac{1}{2}\pi$ satisfy $\cos 4\theta \cos 2\theta = -1$ B1
			11	